Lecture 24 Summary

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1 DC and AC Current-Biased Josephson Junction

Now consider a Josephson junction with an ac current bias in addition to the dc bias. The total current through the JJ is split three ways in general (going through the ideal JJ, resistor, capacitor),
$I_{dc} + I_{ac}\sin(\omega_{ac}t) = I_c \sin\gamma + \frac{\Phi_0}{2\pi R_N}\frac{d\gamma}{dt} + C\frac{\Phi_0}{2\pi}\frac{d^2\gamma}{dt^2}$.
Multiply this current (I) equation by voltage ($\frac{\hbar}{2e}\frac{d\gamma}{dt}$) to get the instantaneous power equation as,
$\frac{d}{dt}\left\{\frac{1}{2}(\frac{\hbar}{2e})^2 C(\frac{d\gamma}{dt})^2 + \left[\frac{-\hbar I}{2e}\gamma - \frac{\hbar I_c}{2e}\cos\gamma\right]\right\} = \left[\frac{\hbar I_{ac}\sin(\omega_{ac}t)}{2e} - (\frac{\hbar}{2e})^2\frac{1}{R_N}\frac{d\gamma}{dt}\right]\frac{d\gamma}{dt}$.
The left side appears to be the time rate of change of kinetic energy plus potential energy, while the right hand side is the power dissipated in the resistor plus a fluctuating force provided by the AC current bias.
The ac bias current has two consequences:
1) The agitation of the washboard produced by the "ac wiggle" will induce the phase particle to jump over the barrier a little early as the washboard is tilted towards the critical current. Thus the critical current will be reduced slightly by this agitation.
2) Shapiro steps. Consider the junction in the running ($\frac{d\gamma}{dt} \neq 0$) finite-voltage state. As it "falls" along the tilted washboard, the fictitious phase particle will speed up and slow down periodically in time. The motion is perioidic, but not sinusoidal. On average the phase particle will cover 2π radians in a period T, resulting in an average angular frequency $\langle\frac{d\gamma}{dt}\rangle = 2\pi/T$.
A resonance condition can be satisfied when the ac drive frequency ω_{ac} coincides with the periodicity of the motion of the phase point $2\pi/T$. In this case the driving current source can "phase lock" with motion of the phase particle and there can be resonant absorption of energy by the JJ. Due to the intrinsic nonlinearity of the JJ, this may happen over a range of dc current values, resulting in a "Shapiro step" in the current-voltage characteristic. The first Shapiro step will occur at a voltage given by $\langle V_1\rangle = \frac{\hbar\omega_{ac}}{2e}$.
As the washboard is tilted further (higher I_{dc}), the phase particle will move faster and it can cover multiple 2π periods of the washboard potential in the

period of the ac drive. Hence there will be higher-order Shapiro steps, given by voltages $\langle V_n \rangle = n\frac{\hbar\omega_{ac}}{2e}$. Plugging in the numbers, the voltage step size will be $V_n = n\frac{\nu}{483.6MHz/\mu V}$, where ν is the linear frequency of the ac current bias in MHz.
The class web site shows these Shapiro steps in the IV curve of a Nb point contact.

2 Shapiro Step Details

Assume that the JJ is voltage biased (difficult to achieve in practice, but it simplifies the calculation considerably) as,
$V(t) = V_{dc} + V_{ac}\cos(\omega_{ac}t)$.
The gauge-invariant phase can be found by integrating the ac Josephson equation $\frac{d\gamma}{dt} = \frac{2e}{\hbar}V(t)$ as,
$\gamma(t) = \gamma(0) + \frac{2e}{\hbar}V_{dc}t + \frac{2e}{\hbar\omega_{ac}}V_{ac}\sin(\omega_{ac}t)$.
Define the Josephson frequency as $\omega_J \equiv \frac{2eV_{dc}}{\hbar}$.
In a typical experiment, one measures the time-averaged current $\langle I \rangle$ as a function of the dc bias voltage V_{dc}. Calculating the current in the RSJ-model junction (ignoring the capacitor) yields,
$I = \frac{V(T)}{R} + I_c \sin\left\{\gamma(0) + \omega_J t + \frac{2eV_{ac}}{\hbar\omega_{ac}}\sin(\omega_{ac}t)\right\}$.
Now use the important identity for the sine of the sine function:
$\sin(a + b\sin\theta) = \sum_{n=-\infty}^{\infty}(-1)^n J_n(b)\sin(a - n\theta)$ to find
$I(t) = \frac{V(T)}{R} + I_c\sum_{n=-\infty}^{\infty}(-1)^n J_n(\frac{2\pi V_{ac}}{\Phi_0\omega_{ac}})\sin\left[\gamma(0) + (\omega_J - n\omega_{ac})t\right]$.
Consider the time average of the current $\langle I \rangle$, which is the quantity usually measured in experiment. At an arbitrary driving frequency ω_{ac} the sine term will average to zero. However, at the special frequencies $\omega_J = n\omega_{ac}$ there will be a non-zero result,
$\langle I \rangle = \frac{V(T)}{R} + I_c\sum_{n=-\infty}^{\infty}(-1)^n J_n(\frac{2\pi V_{ac}}{\Phi_0\omega_{ac}})\sin\left[\gamma(0)\right]\delta_{\omega_J, n\omega_{ac}}$.
These dc voltage values produce a range of currents depending on the value of $\gamma(0)$, creating a series of spikes periodic in V_{dc} riding on top of an Ohmic background. These are the Shapiro steps.
The above calculation predicts that the widths of the steps will be modulated with ac voltage amplitude V_{ac}. Due to the dependence of the Bessel functions for small arguments, $J_n(x) \sim x^n$ each step will appear in order as the microwave power is increased. The widths of the steps will change in a non-monotonic manner with increasing V_{ac}. The NIST voltage standard is based on a giant Shapiro step.

3 DC SQUIDs

A SQUID is a Superconducting QUantum Interference Device. The DC SQUID uses two superconductors connected by two Josephson junctions. It acts as a sensitive magnetic flux to voltage transducer. The dc SQUID is current biased,

and the voltage drop on the device is monitored as a function of magnetic flux in the SQUID loop.
The bias current splits two ways and can be written as,
$I_b = I_c \sin\gamma_1 + I_c \sin\gamma_2$, where it is assumed that both junctions have identical parameters (I_c, R, C), but their GIPD are different in general. Using a trigonometric identity, one can write the bias current as,
$I_b = 2I_c \cos(\frac{\gamma_1-\gamma_2}{2})\sin(\frac{\gamma_1+\gamma_2}{2})$.
Now we insist that the phase of the macroscopic quantum wavefunction be the same, modulo 2π upon completing a circuit through the SQUID loop and coming back to the same point. Using the expressions for the GIPD at the two junctions, and the London relation between the current density, vector potential and gradient of the phase in the superconductors, one can derive the following result:
$\gamma_2 - \gamma_1 = 2\pi(n + \frac{\Phi}{\Phi_0})$, where $n = 0, \pm 1, \pm 2, ...$ and Φ is the magnetic flux in the entire SQUID loop.
One can use this to write the sum of the GIPDs as $\frac{\gamma_1+\gamma_2}{2} = \gamma_1 + \pi(n + \frac{\Phi}{\Phi_0})$.
With application of another trigonometric identity, we arrive at the result,

$$I_b = 2I_c \cos(\pi\frac{\Phi}{\Phi_0})\sin(\gamma_1 + \pi\frac{\Phi}{\Phi_0}) = \widetilde{I}_c \sin\widetilde{\gamma}.$$

In other words, the dc SQUID acts as a single Josephson junction with a flux-tunable critical current. The renormalized critical current is $\widetilde{I}_c(\Phi) = 2I_c\cos(\pi\frac{\Phi}{\Phi_0})$, the renormalized phase is given by $\widetilde{\gamma} = \gamma_1 + \pi\frac{\Phi}{\Phi_0}$, the renormalized resistance is $\widetilde{R} = R/2$, and the renormalized capacitance is given by $\widetilde{C} = 2C$.
(Note that in this derivation we assume that the "self-flux" produced by screening currents in the loop is small, or in other words $LI_c << \Phi_0$, where L is the self-inductance of the loop.)

The critical current of the SQUID is a periodic function of flux, repeating every time Φ advances through Φ_0. It ranges in value from $2I_c$ to zero periodically in flux. Consider the case of a SQUID with small capacitance. It will have an I-V curve given by,

$$\langle V\rangle = \begin{cases} 0 & I < \widetilde{I}_c \\ \widetilde{R}_N\sqrt{I^2 - \widetilde{I}_c^2(\Phi)} & I > \widetilde{I}_c \end{cases}$$

where $\widetilde{I}_c$ can be modulated between $2I_c$ and 0, depending on the flux applied to the SQUID. If we now bias the SQUID with a current just under $2I_c$, the voltage developed on the SQUID will be a function of flux applied. The dependence will be periodic in flux with period Φ_0, but not sinusoidal. The transfer function between voltage and flux is nonlinear, but can be linearized for small ranges of applied flux.